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Remark on the Strichartz estimates in the radial case

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We prove up to an endpoint the full radial Strichartz estimates for the Schrödinger-type equation in the radial case. Our results improve the one obtained in Guo and Wang (2011) [2]. The main ingredient is that we exploit some orthogonality to overcome the logarithmic divergence.

1. Introduction

In this paper, we consider the following Cauchy problem

$$iu_t + |\nabla|^a u = 0, \quad u(x,0) = u_0(x) \tag{1.1}$$

where $u(x,t): \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$, $0 < a \neq 1$, $n \geqslant 2$, and $\mathcal{F}(|\nabla|^a f) = |\xi|^a \mathcal{F}(f)$. We assume that $u_0 \in \dot{H}^s(\mathbb{R}^n)$ and u_0 is spherically symmetric. The main purpose is to study the following Strichartz-type estimates for the solution of (1.1): $k \in \mathbb{Z}$

$$\left\| e^{it|\nabla|^a} P_k u_0 \right\|_{L_t^q L_x^r} \leqslant C 2^{k(\frac{n}{2} - \frac{a}{q} - \frac{n}{r})} \|u_0\|_{L^2}, \tag{1.2}$$

where we denote $e^{it|\nabla|^a} u_0 = \mathcal{F}^{-1} e^{it|\xi|^a} \mathcal{F} u_0$, and P_k is the Littlewood–Paley projector to the frequency $\{|\xi| \sim 2^k\}$ (which will be defined in the end of this section). The main result of this paper is

Theorem 1.1. *Let $0 < a \neq 1$, $n \geqslant 2$ and $k \in \mathbb{Z}$. Assume $2 \leqslant q \leqslant \infty$, $\frac{2}{q} + \frac{2n-1}{r} \leqslant n - \frac{1}{2}$ and $(q,r) \neq (2, \frac{4n-2}{2n-3})$. Then there exists $C > 0$ such that* (1.2) *holds for all radial functions $u_0 \in L^2$.*

The Strichartz estimates (1.2) for the Schrödinger-type equation (Schrödinger equation corresponds to $a = 2$) have been extensively studied and play fundamental roles. For the general non-radial Strichartz estimates we refer the readers to [11] and the references therein. In the radial case, (1.2) holds for a wider range of (q,r), see [10,3–7,2]. The sharp range of (1.2) for $a = 1$ was known, see [10,1,5]. On the other hand, the sharp range for $a \neq 1$ is not known so far. The most generalized range was obtained in [2] which states that (1.2) holds if (q,r) satisfies the conditions in Theorem 1.1 and moreover $q \geqslant r$ on the boundary $\frac{2}{q} + \frac{2n-1}{r} = n - \frac{1}{2}$. Thus Theorem 1.1 improves the results in [2] by removing the condition $q \geqslant r$ on the boundary except the endpoint $(q,r) = (2, \frac{4n-2}{2n-3})$ which is the only unknown point for (1.2) in view of the counter-example given in [2].

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To prove Theorem 1.1, the key point is to exploit some orthogonality. It was known that on the boundary $\frac{2}{q}+\frac{2n-1}{r}=n-\frac{1}{2}$, there is logarithmic divergence on summing over all the dyadic pieces. In [2], the authors used double-weight Hardy–Littlewood inequality for $q=r$ to overcome the logarithmic divergence. In this paper, we prove some almost-orthogonality when $q=r$ between dyadic pieces, then interpolating with the endpoint $(q,r)=(2,\frac{4n-2}{2n-3})$, we can obtain almost-orthogonality between the two points (which explains why we can't cover the endpoint). By this orthogonality, we could sum over all dyadic pieces.

Throughout this paper, we will use C to denote positive universal constants, which can be different at different places. $A\lesssim B$ means that $A\leqslant CB$, and $A\sim B$ stands for $A\lesssim B$ and $B\lesssim A$. We use $\hat{f}(\xi)$ and $\mathcal{F}(f)$ to denote the spatial Fourier transform of f on $\mathbb{R}^n$ defined by

$$\hat{f}(\xi)=\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx.$$

We denote by p' the dual number of $p\in[1,\infty]$, i.e., $1/p+1/p'=1$. Let $\eta(x):\mathbb{R}\to[0,1]$ be a smooth even function such that $\operatorname{supp}(\eta)\subseteq\{x\colon\ |x|\leqslant 2\}$, and $\eta(x)\equiv 1$ if $|x|\leqslant 1$. Let $\psi(x)=\eta(x)-\eta(2x)$, and P_k be the Littlewood–Paley projector for $k\in\mathbb{Z}$, namely $P_kf=\mathcal{F}^{-1}\psi(2^{-k}|\xi|)\mathcal{F}f$.

We will use Lebesgue spaces $L^p:=L^p(\mathbb{R}^n)$, $\|\cdot\|_p:=\|\cdot\|_{L^p}$ and the space–time norm $L_t^qL_x^r$ of f on $\mathbb{R}\times\Omega$ by

$$\big\|f(t,x)\big\|_{L_t^qL_x^r(\mathbb{R}\times\Omega)}=\Big\|\big\|f(t,x)\big\|_{L_x^r(\Omega)}\Big\|_{L_t^q(\mathbb{R})},$$

where $\Omega\subset\mathbb{R}^n$. When $q=r$, we abbreviate it by $L_{t,x}^q(\mathbb{R}\times\Omega)$. For $j\in\mathbb{Z}$, denote

$$A_j:=\big\{x\in\mathbb{R}^n\colon\ 2^{j-1}\leqslant|x|<2^j\big\},\qquad I_j=\big[2^{j-1},2^j\big).$$

2. Proof of Theorem 1.1

The estimate (1.2) is reduced in [2] to a one-dimensional problem. For the completeness, we repeat this reduction. From the definition of P_k we see that P_ku_0 is radial when u_0 is radial. By a scaling transform: for $j\in\mathbb{Z}$

$$u_0(x)\to u_{0,j}=u_0\big(2^jx\big),\qquad e^{it|\nabla|^a}P_ku_0\to e^{it|\nabla|^a}P_ku_{0,j}$$

we get that (1.2) is equivalent to the case $k=0$, namely

$$\big\|e^{it|\nabla|^a}P_0u_0\big\|_{L_t^qL_x^r}\leqslant C\|u_0\|_{L^2}. \tag{2.1}$$

We will prove (2.1) under the conditions in Theorem 1.1. In view of the results in [2], we only need to prove (2.1) for (q,r) satisfying

$$2\leqslant q\leqslant\frac{4n+2}{2n-1},\qquad \frac{2}{q}+\frac{2n-1}{r}=n-\frac{1}{2}, \tag{2.2}$$

$$(q,r)\neq\left(2,\frac{4n-2}{2n-3}\right). \tag{2.3}$$

It is well known that if $f(x)=g(|x|)$ is radial, then the Fourier transform of f is also radial (cf. [8]), and

$$\hat{f}(\xi)=2\pi\int_0^\infty g(s)s^{n-1}\big(s|\xi|\big)^{-\frac{n-2}{2}}J_{\frac{n-2}{2}}\big(s|\xi|\big)\,ds. \tag{2.4}$$

Thus if $\widehat{u_0}(\xi)=h(|\xi|)$ is radial, then $e^{it|\nabla|^a}P_0u_0=F(t,|x|)$, and

$$F\big(t,|x|\big)=2\pi\int_0^\infty e^{its^a}\psi(s)h(s)s^{n-1}\big(s|x|\big)^{-\frac{n-2}{2}}J_{\frac{n-2}{2}}\big(s|x|\big)\,ds. \tag{2.5}$$

Here $J_m(r)$ denote the Bessel function:

$$J_m(r)=\frac{(r/2)^m}{\Gamma(m+1/2)\pi^{1/2}}\int_{-1}^1 e^{irt}\big(1-t^2\big)^{m-1/2}\,dt,\quad m>-1/2.$$

Thus (2.1) is equivalent to

$$\left\| F\big(t, |x|\big) \right\|_{L_t^q L_x^r} \leqslant C \|h\|_{L^2}. \tag{2.6}$$

We first list some properties of $J_m(r)$ that will be used in the following lemma. For their proof we refer the readers to [9].

Lemma 2.1 *(Properties of the Bessel function). We have for* $0 < r < \infty$ *and* $m > -\frac{1}{2}$:

(i) $J_m(r) \leqslant C r^m$,
(ii) $J_m(r) \leqslant C r^{-\frac{1}{2}}$,
(iii) $\frac{d}{dr}(r^{-m} J_m(r)) = -r^{-m} J_{m+1}(r)$.

Decompose $F(t, |x|) = F_{\leqslant 1}(t, |x|) + F_{\geqslant 2}(t, |x|)$, where $F_{\leqslant 1}(t, |x|) = F(t, |x|) 1_{|x| \leqslant 2}$ and $F_{\geqslant 2} = F - F_{\leqslant 1}$.

Lemma 2.2. *Assume* $0 < a \neq 1$, $n \geqslant 2$ *and* $j \in \mathbb{Z}$. *Then*

$$\left\| F\big(t, |x|\big) \right\|_{L_t^2 L_x^{\frac{2n}{n-2}}(\mathbb{R} \times A_j)} \leqslant C \|h\|_{L^2}. \tag{2.7}$$

Proof. $(2, \frac{2n}{n-2})$ is the classical endpoint, see [11] for $n \geqslant 3$. For $n = 2$ and $a = 2$, see [12]. We need to show the case $0 < a \neq 1$ and $n = 2$. For $j \geqslant 1$, using Sobolev's inequality and Lemma 2.1(ii), (iii), we have

$$\left\| F(t,r)\psi_j(r) \right\|_{L_t^2 L_r^\infty} \lesssim \left\| F(t,r)\psi_j(r) \right\|_{L_t^2 L_r^2} + \left\| \frac{d}{dr}\big(F(t,r)\psi_j(r)\big) \right\|_{L_t^2 L_r^2} \lesssim \|h\|_2.$$

For $j \leqslant 1$, similarly, we have

$$\left\| F(t,r)\psi_j(r) \right\|_{L_t^2 L_r^\infty} \lesssim \left\| F(t,r)\eta(r) \right\|_{L_t^2 L_r^2} + \left\| \frac{d}{dr}\big(F(t,r)\eta(r)\big) \right\|_{L_t^2 L_r^2} \lesssim \|h\|_2.$$

This completes the proof of the lemma. □

Using Lemma 2.2, we see that $\|F_{\leqslant 1}\|_{L_t^q L_x^r} \lesssim \|h\|_2$ for (q, r) satisfying the conditions in Theorem 1.1. It remains to control $F_{\geqslant 2}$. As in [2], divide $F_{\geqslant 2}$ into two parts: the main term and the error term, namely

$$F_{\geqslant 2}\big(t, |x|\big) = M\big(t, |x|\big) 1_{|x| \geqslant 2} + E\big(t, |x|\big) 1_{|x| \geqslant 2} \tag{2.8}$$

with

$$M(t,r) = c_n r^{-\frac{n-1}{2}} \int_{\mathbb{R}} \psi_0(s) h(s) s^{\frac{n-1}{2}} e^{i(rs - ts^a)}\, ds + \bar{c}_n r^{-\frac{n-1}{2}} \int_{\mathbb{R}} \psi_0(s) h(s) s^{\frac{n-1}{2}} e^{-i(rs + ts^a)}\, ds,$$

$$E(t,r) = c_1 \int_{\mathbb{R}} \psi_0(s) h(s) s^{n-1} e^{-its^a - irs} E_+(rs)\, ds - c_2 \int_{\mathbb{R}} \psi_0(s) h(s) s^{n-1} e^{-its^a + irs} E_-(rs)\, ds.$$

We recall the estimates in [2] for the main term and error term where M_1 denotes the first term in M and $M_2 = M - M_1$.

Lemma 2.3. *(See [2, Lemmas 2.11, 2.12].) Assume* $0 < a \neq 1$, $n, j \geqslant 2$ *and* $2 \leqslant r \leqslant \frac{2n}{n-2}$. *Then*

$$\left\| M_k\big(t, |x|\big) \right\|_{L_t^2 L_x^r(\mathbb{R} \times A_j)} \lesssim 2^{j(\frac{2n-1}{2r} - \frac{2n-3}{4})} \|h\|_{L^2}, \quad k = 1, 2, \tag{2.9}$$

$$\left\| E\big(t, |x|\big) \right\|_{L_t^2 L_x^r(\mathbb{R} \times A_j)} \lesssim 2^{-\frac{j}{2}(\frac{n}{r} - \frac{n-2}{2})} \|h\|_{L^2}. \tag{2.10}$$

Using the lemma above $r = \frac{4n-2}{2n-3}$ and interpolation with the trivial point $(\infty, 2)$, we see that for (q, r) satisfying (2.2),

$$\|E\|_{L_t^q L_x^r(\mathbb{R} \times A_j)} \lesssim 2^{-j\theta} \|h\|_2$$

for $j \geqslant 2$ and some $\theta > 0$, and

$$\|M_k\|_{L_t^q L_x^r(\mathbb{R} \times A_j)} \lesssim \|h\|_2, \quad k = 1, 2. \tag{2.11}$$

Thus we see that when (q,r) satisfies (2.2), one just fails to sum over $j \geqslant 2$ for the term M. To overcome this logarithmic divergence, we need to exploit some orthogonality. To prove Theorem 1.1, it suffices to show

$$\left\| M_k\big(t,|x|\big)1_{|x|\geqslant 2}\right\|_{L_t^q L_x^r} \lesssim \|h\|_2, \quad k=1,2, \tag{2.12}$$

if (q,r) satisfies (2.2)–(2.3). By symmetry we only need to consider $k=1$.

For $j \in \mathbb{Z}$ and $j \geqslant 2$, we define the operator T_j

$$T_j(f)(t,\rho) = \chi_{I_j}(\rho)|\rho|^{-\frac{n-1}{2}} \int_{\mathbb{R}} \psi(s) f(s) e^{i(\rho s - t s^a)}\, ds.$$

The dual operator T_j^* is

$$T_j^*(g)(s) = \psi(s) \int \chi_{I_j}(\rho) g(t,\rho)|\rho|^{-\frac{n-1}{2}} e^{-i(\rho s - t s^a)}\, dt\, d\rho.$$

Then (2.11) means

$$\left\| \rho^{\frac{n-1}{r}} T_j(f)(\rho)\right\|_{L_t^q L_\rho^r} \leqslant C\|f\|_2. \tag{2.13}$$

To prove (2.12), it suffices to show

$$\left\| \rho^{\frac{n-1}{r}} \sum_{j\geqslant 2} T_j \right\|_{L^2 \to L_t^q L_\rho^r} \leqslant C. \tag{2.14}$$

By TT^* method, it is equivalent to show

$$\left\| \rho^{\frac{n-1}{r}} \sum_{j,k\geqslant 2} T_j T_k^*\big(\rho^{\frac{n-1}{r}} f\big)\right\|_{L_t^q L_\rho^r} \leqslant C\|f\|_{L_t^{q'} L_\rho^{r'}}. \tag{2.15}$$

To overcome the logarithmic divergence, we need to use the following crucial almost-orthogonality property of T_j:

Lemma 2.4. *Assume $n \geqslant 2$ and (q,r) satisfies* (2.2)–(2.3)*, $j,k \geqslant 2$. Then*

$$\left\| \rho^{\frac{n-1}{r}} T_j T_k^*\big(\rho^{\frac{n-1}{r}} f\big)\right\|_{L_t^q L_\rho^r} \lesssim 2^{-|j-k|(\frac{n}{q}-\frac{n-1}{2})(\frac{1}{2}-\frac{1}{q})(2n+1)} \|f\|_{L_t^{q'} L_\rho^{r'}}. \tag{2.16}$$

Proof. If j, k satisfies $|j-k| \leqslant 5$, then the lemma follows from (2.13). Thus we assume $|j-k| \geqslant 5$. For $(q,r) = (2, \frac{4n-2}{2n-3})$, then (2.16) also follows from (2.13). By Riesz–Thorin interpolation, it suffices to show that (2.16) holds for $q=r=\frac{4n+2}{2n-1}$. Fix a Schwarz function $f \in L_{t,\rho}^{q'}$ with $\|f\|_{L_{t,\rho}^{q'}} \leqslant 1$. Then

$$\begin{aligned} T_j T_k^* f &= \chi_{I_j}(\rho)\rho^{-\frac{n-1}{2}} \int \psi^2(s)\chi_{I_k}\big(\rho'\big) f\big(t',\rho'\big)\rho'^{-\frac{n-1}{2}} e^{i((\rho-\rho')s-(t-t')s^a)}\, ds\, dt'\, d\rho' \\ &= \chi_{I_j}(\rho)\rho^{-\frac{n-1}{2}} \int K\big(\rho-\rho', t-t'\big) f\big(t',\rho'\big)\chi_{I_k}\big(\rho'\big)\rho'^{-\frac{n-1}{2}}\, dt'\, d\rho' \end{aligned}$$

where

$$K(x,y) = \int_{\mathbb{R}} \psi^2(s) e^{i(xs - ys^a)}\, ds.$$

By the Van der Corput lemma (see [9]), and the condition $0 < a \neq 1$, we have

$$\big|K(x,y)\big| \lesssim |x|^{-1/2}.$$

On the other hand, by Plancherel's equality after making a change of variable $\tau = s^a$, we have

$$\left\| \int K\big(x, y-y'\big) g\big(y'\big)\, dy'\right\|_{L_y^2} \lesssim \|g\|_{L^2}.$$

Thus by interpolation, we obtain

$$\left\| \int K\big(\rho-\rho', t-t'\big) f\big(t',\rho'\big)\, dt'\right\|_{L_t^q} \lesssim \big|\rho-\rho'\big|^{-(\frac{1}{2}-\frac{1}{q})} \big\|f\big(\cdot,\rho'\big)\big\|_{L_t^{q'}}.$$

Using Minkowski's inequality and Hölder's inequality, we get

$$\begin{aligned}\big\|\rho^{\frac{n-1}{q}} T_j T_k^*\big(\rho^{\frac{n-1}{q}} f\big)\big\|_{L^q_{t,\rho}} &\lesssim \left\|\chi_{I_j}(\rho)\rho^{\frac{n-1}{q}-\frac{n-1}{2}} \int \big|\rho-\rho'\big|^{-(\frac{1}{2}-\frac{1}{q})} \big\|f\big(\cdot,\rho'\big)\big\|_{L^{q'}_t} \chi_{I_k}\big(\rho'\big)\rho'^{\frac{n-1}{q}-\frac{n-1}{2}} d\rho'\right\|_{L^q_\rho} \\ &\lesssim 2^{(j+k)(\frac{n}{q}-\frac{n-1}{2})} 2^{-j(\frac{1}{2}-\frac{1}{q})} \|f\|_{L^{q'}_t L^{r'}_\rho} \\ &\lesssim 2^{(k-j)(\frac{n}{q}-\frac{n-1}{2})} \|f\|_{L^{q'}_t L^{r'}_\rho}\end{aligned}$$

which completes the proof of the lemma. □

We are ready to prove (2.15). By the support properties of T_j and the fact $q, r \geqslant 2$, we have

$$\left\|\rho^{\frac{n-1}{r}} \sum_{j,k\geqslant 2} T_j T_k^*\big(\rho^{\frac{n-1}{r}} f\big)\right\|_{L^q_t L^r_\rho} \lesssim \left(\sum_{j\geqslant 2}\left\|\rho^{\frac{n-1}{r}} \sum_{k\geqslant 2} T_j T_k^*\big(\rho^{\frac{n-1}{r}} f\big)\right\|^2_{L^q_t L^r_\rho}\right)^{1/2}. \tag{2.17}$$

Let $\beta = (\frac{n}{q} - \frac{n-1}{2})(\frac{1}{2} - \frac{1}{q})(2n+1)$. We get that $\beta > 0$ if (q, r) satisfies (2.2)–(2.3). Applying Lemma 2.4, Young's inequality and Minkowski's inequality, we get

$$(2.17) \lesssim \left\|\sum_{k\geqslant 2} 2^{-|j-k|\beta} \big\|\chi_{I_k}(\rho) f\big\|_{L^{q'}_t L^{r'}_\rho}\right\|_{l^2_j} \lesssim \big\|\chi_{I_k}(\rho) f\big\|_{l^2_k L^{q'}_t L^{r'}_\rho} \lesssim \|f\|_{L^{q'}_t L^{r'}_\rho}$$

which complete the proof of Theorem 1.1.

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